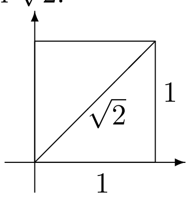
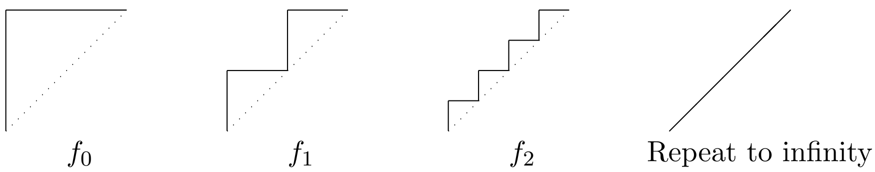
Trollmath explained

The confusing proof

Let us consider the unit square in the interval $[0,1]$ with the diagonal of length $\sqrt{2}$:



This diagonal can be approximated by the sequence f_n of functions:



Since the arc length of f_n is 2 for all $n \in \mathbb{N}$, we conclude that $2 = \sqrt{2}$.

What does this proof get right?

For $n \to \infty$, the area A_{f_n} below f_n converges to $\frac{1}{2}$, which is equal to the area below the diagonal of the unit square. If you look at the graphs, you see that this is just a Riemann Integral.

$$A_{f_n} = \sum_{i=1}^{2^n} \frac{i}{2^n} \frac{1}{2^n} = \frac{1}{2^{2n}} \sum_{i=1}^{2^n} i = \frac{1}{2^{2n}} \frac{2^n(2^n+1)}{2} = \frac{1}{2^{2n}} \frac{2^{2n}+2^n}{2} = \frac{1}{2}(1+2^{-n})$$

$$\lim_{n\to\infty} A_{f_n} = \lim_{n\to\infty} \frac{1}{2}(1+2^{-n}) = \frac{1}{2}$$

Furthermore, we can show that it's actually true that the sequence f_n of functions converges uniformly (and thus also pointwise) to the function $y = x$.

$$\lim_{n\to\infty} \sup_{x\in[0,1]} |f_n(x) - x| = \lim_{n\to\infty} \frac{1}{2^n} = 0$$

This equation holds because $|f_n(x) - x| \leq \frac{1}{2^n}$.

There is no infinitesimally small zig-zag, no fractal, it converges exactly to the diagonal $y = x$ of the unit square.

What's wrong with this proof?

Obviously, there has to be something wrong with this proof. To deduce that in the interval $[0, 1]$, the arc length of $y = x$ is equal to the arc length of the functions f_n because they converge uniformly to $y = x$, is wrong.

Pointwise and uniform convergence are not strong enough criteria to make such a claim about arc length. For this, we need an even stronger notion of limit, for example, if we restrict ourselves to differentiable curves (or piecewise differentiable curves) and require that not only the functions converge uniformly, but that their derivatives do as well. Note that this is a *sufficient*, but not a *necessary* condition. Arc length can be defined on non-differentiable functions. In fact in general, the notion of differentiability is not defined on a metric space.